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# STRESS DISTRIBUTION NEAR THE EDGE OF A CRACK IN A PRESTRESSED ELASTIC BODY*

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The effect of preliminary loading in the plane of a crack on the stress distribution around the crack edge is investigated for normal fracture, as well as transverse and longitudinal shear. Unlike /1/ where complex potentials of the linearized theory of elasticity were used, a different method is employed to solve the problem and the following questions are discussed: the effect of initial deformation on the stress intensity coefficients and the connection between the conditions of solvability of the problem of a crack and the Hadamard inequality. In the case of an isotropic incompressible material of general type it is shown that the initial deformation does not affect the order of the stress singularity at the crack edge. Asymptotic representations of the displacements and stresses near the crack edge are obtained in more explicit form than those in /1/. It is established that the initial deformation does not influence the stress intensity coefficients of the normal fracture and longitudinal shear, but increases the stress intensity coefficient in the case of transverse shear.

We consider an unbounded elastic space weakened by a plane, infinitely thin crack (slit) bounded by a small closed contour. We assume that the elastic medium undergoes a homogeneous finite deformation such that the planes parallel to the plane of the crack are stress-free.

We shall consider the problem of superimposing, on the finite homogeneous deformation described above, a small deformation caused by a uniform loading applied to the crack surface. Since the additional deformation is assumed small, we shall consider the latter problem in the linearized formulation.

We note that superposition of the solutions of the problem formulated and of the problem of small homogeneous deformation in a prestressed space without cracks, appearing as a result of a uniform applied load, yields a solution of the problem of the deformation of a body with a load-free crack by forces applied at infinity.

In order to study the asymptotic distribution of the displacements, deformation and stresses near the crack contour, it is sufficient, according to the microscope principle /2/, a study three two-dimensional problems for a rectilinear crack, namely the problem of normal fracture, of transverse shear and of longitudinal shear /2, 3/:

Without the mass forces the linearized equations of equilibrium have the form /4/

$$\frac{\partial \theta_{sk}}{\partial x_s} = 0, \quad \theta_{sk} = \sigma_{sk} - t_{mk}\frac{\partial u_s}{\partial x_m} + t_{sk}\frac{\partial u_m}{\partial x_m} \tag{1}$$

where $x_s$ $(s = 1, 2, 3)$ are the Cartesian coordinates in the predeformed state, $u_m$ are the components of additional displacement, $t_{mk}$ are the initial stresses and $\sigma_{sk}$ are the additional true stresses, i.e. those caused by the additional displacements in the increments of the Cauchy stress tensor components. In the case of an isotropic incompressible body, when the coordinate axes coincide with the principle axes of the initial homogeneous deformation, the quantities $\sigma_{sk}$ are given by /5, 6/:

$$\sigma_{mn} = s_{mn} - t_{mk}\omega_{kn} + \omega_{mk}t_{kn} \tag{2}$$

$$s_{11} = (\lambda_1^2\Pi_{11} + \lambda_1\Pi_1)\,\varepsilon_{11} + \lambda_1\lambda_2\Pi_{12}\varepsilon_{22} + \lambda_1\lambda_3\Pi_{13}\varepsilon_{33} + p \quad (1\ 2\ 3)$$

$$s_{12} = s_{21} = \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 - \lambda_2^2}(\lambda_1\Pi_1 - \lambda_2\Pi_2)\,\varepsilon_{12} \quad (1\ 2\ 3)$$

$$\Pi_i = \frac{\partial \Pi}{\partial \lambda_i}, \quad \Pi_{ij} = \frac{\partial^2 \Pi}{\partial \lambda_i \partial \lambda_j}$$

$$\varepsilon_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right), \quad \omega_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} - \frac{\partial u_n}{\partial x_m}\right)$$

Here $\lambda_i$ $(i = 1, 2, 3)$ are the initial extensions in the initial deformed state, $\Pi = \Pi(\lambda_1, \lambda_2, \lambda_3)$ is the specific potential energy of the material, $p$ is the additional pressure, and the

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symbol (123) denotes that the unwritten relations are obtained by cyclic permutation of the indices.

We shall consider the problem of a rectilinear crack with the edge parallel to the principal axis of the initial deformation. In this case the problem of normal fracture, transverse shear and longitudinal shear can be considered independently and the first two problems are the problems of plane deformation, while in the third problem we have antiplane deformation.

Let the trace of the plane crack with edges parallel to the $x_2$ axis in the $x_1x_2$ plane coincide with the segment of the abscissa $|x_1| \leqslant a$. For the plane additional deformation we write the system of linearized equations of equilibrium and conditions of incompressibility in the form

$$\mu\frac{\partial^2 u}{\partial x^2}+\nu\frac{\partial^2 u}{\partial y^2}+\nu\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial q}{\partial x}=0 \tag{3}$$

$$\nu\frac{\partial^2 u}{\partial x\partial y}+\varkappa\frac{\partial^2 v}{\partial x^2}+\frac{\partial q}{\partial y}=0,\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$x=x_1,\quad y=x_2,\quad u=u_1,\quad v=u_2,\quad \theta_{22}=\sigma_{22}=q$$

$$\mu=2\lambda_2\Pi_2+\lambda_1^2\Pi_{11}+\lambda_2^2\Pi_{22}-2\lambda_1\lambda_2\Pi_{12}$$

$$\varkappa=\frac{\lambda_1^2(\lambda_1\Pi_1-\lambda_2\Pi_2)}{\lambda_1^2-\lambda_2^2},\quad \nu=\frac{\lambda_2^2(\lambda_1\Pi_1-\lambda_2\Pi_2)}{\lambda_1^2-\lambda_2^2}$$

The problem of normal fracture crack whose edges are under a uniform pressure $p_0$, is equivalent to the problem for a half-plane $y \geqslant 0$ with the following boundary conditions at the boundary $y = 0$:

$$\theta_{21}=\nu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0,\quad |x|<\infty \tag{4}$$

$$q=-p_0,\ |x|<a;\ v=0,\ |x|>a \tag{5}$$

Applying the integral Fourier transform to (3)–(5), we arrive at a paired integral equation whose solution is known /7/. The displacements and additional non-symmetric stresses in the half-plane $y \geqslant 0$ are given by the formulas

$$u=-p_0\Phi\,\mathrm{Im}\,\{(\omega_1-\omega_2)^{-1}[\omega_1(1+\omega_2^2)(\sqrt{a^2-z_1^2}+iz_1)-\ldots]\} \tag{6}$$

$$v=-p_0\Phi\,\mathrm{Re}\,\{(\omega_1-\omega_2)^{-1}[(1+\omega_2^2)(\sqrt{a^2-z_1^2}+iz_1)-\ldots]\}$$

$$\Phi=[(1+\omega_1^2)(1+\omega_2^2)\nu+(\omega_1\omega_2-1)\mu]^{-1}$$

$$q=-p_0\Phi\,\mathrm{Re}\left\{(\omega_1-\omega_2)^{-1}\left[\omega_1(1+\omega_2^2)(\nu-\mu+\nu\omega_1^2)\left(1+\frac{iz_1}{\sqrt{a^2-z_1^2}}\right)-\ldots\right]\right\}$$

$$\theta_{11}=-p_0\Phi(1+\omega_1^2)(1+\omega_2^2)\nu\,\mathrm{Re}\left\{(\omega_1-\omega_2)^{-1}\times\left[\omega_1\left(1+\frac{iz_1}{\sqrt{a^2-z_1^2}}\right)-\ldots\right]\right\}$$

$$\theta_{12}=p_0\Phi\,\mathrm{Im}\left\{(\omega_1-\omega_2)^{-1}\left[(\varkappa+\nu\omega_1^2)(1+\omega_2^2)\left(1+\frac{iz_1}{\sqrt{a^2-z_1^2}}\right)-\ldots\right]\right\}$$

$$\theta_{21}=p_0\Phi(1+\omega_1^2)(1+\omega_2^2)\nu\,\mathrm{Im}\left\{(\omega_1-\omega_2)^{-1}i\left[\frac{z_1}{\sqrt{a^2-z_1^2}}-\ldots\right]\right\}$$

Here

$$z_1=x+\omega_1 iy,\ z_2=x+\omega_2 iy \tag{7}$$

where $\omega_1$, $\omega_2$ are roots of the equation

$$\nu\omega^4-(\mu-2\nu)\omega^2+\varkappa=0 \tag{8}$$

with positive real part. Repeated dots denote the expression obtained from the first term within the square brackets by interchanging the indices 1 and 2.

We can show, as in /8/, that when the stricter Hadamard inequality is satisfied, Eq.(8) is guaranteed to have two roots with positive real parts, and two roots with negative real parts. This in turn ensures the existence of solutions of the problem (3)–(5), decaying as $y\to\infty$.

On the extension of the crack line, the displacements and normal stress have, at $y=0$, the form

$$u|_{y=0}=p_0\Omega^{-1}(\omega_1\omega_2-1)\left\{\begin{array}{ll}x, & |x|\leqslant a\\ (x-\sqrt{x^2-a^2}, & x\geqslant a\end{array}\right. \tag{9}$$

$$v|_{y=+0} = \begin{cases} p_0\Omega^{-1}(\omega_1 + \omega_2)\sqrt{a^2 - x^2}, & |x| \leqslant a \\ 0, & |x| \geqslant a \end{cases}$$

$$\Omega = (1 + \omega_1^2)(1 + \omega_2^2)\nu + (\omega_1\omega_2 - 1)\mu$$

$$q|_{y=0} = \begin{cases} -p_0, & |x| < a \\ p_0(|x|/\sqrt{x^2 - a^2} - 1), & |x| > a \end{cases}$$

Note that (9) can be used when (8) has multiple roots.

In order to study the form of the solution (6) near the slit end $x = a$, we introduce polar coordinates with origin at the point $x = a$, $y = 0$:

$$z = x + iy = a(1 + \rho e^{i\varphi}),\ x = a(1 + \rho\cos\varphi),\ y = \rho\sin\varphi$$

We shall assume that $-\pi \leqslant \varphi \leqslant \pi$. Let us denote by $\tau_k$ and $\chi_k$ the real and imaginary parts of the roots $\omega_k$, i.e.

$$\omega_k = \tau_k + i\,\chi_k\ (k = 1, 2) \tag{10}$$

(if the roots are complex, then $\tau_1 = \tau_2$, $\chi_1 = -\chi_2$).

Let us write

$$z_k = a(1 + \rho_k e^{i\varphi_k})\ (k = 1, 2) \tag{11}$$
$$\rho_k \geqslant 0,\ -\pi \leqslant \varphi_k \leqslant \pi$$

From (7), (11) we obtain, for $k = 1, 2$

$$\begin{aligned} \rho_k &= \rho\,[\cos^2\varphi - \chi_k\sin 2\varphi + (\tau_k^2 + \chi_k^2)\sin^2\varphi]^{1/2} \\ \rho_k\sin\varphi_k &= \rho\tau_k\sin\varphi \\ \rho_k\cos\varphi_k &= \rho(\cos\varphi - \chi_k\sin\varphi) \end{aligned} \tag{12}$$

Substituting expressions (11) into (6) and taking into account formulas (12), we obtain expressions for the displacements and stresses as functions of the polar coordinates $\rho$, $\varphi$. Retaining in these expressions terms of the lowest order in $\rho$, only, we obtain the asymptotic representations for the displacement and stress fields near the crack edge. The representations become very bulky in the general case, and will not therefore be given. The form of the solution near the crack tip has been studied in detail for Mooney material in /9/.

From (6), (11), (12) and (9) it follows naturally that for any material belonging to the class of isotropic incompressible bodies in question, satisfying the stronger Hadamard inequality, the displacements near the crack edge are of the order of $\rho^{1/2}$, and the stresses are of the order of $\rho^{-1/2}$. Thus the initial stresses acting in the plane of the crack do not affect the order of the stress singularities near the crack edge.

In the problem of transverse shear the crack edges are loaded by a uniform tangential load of strength $\tau_0$, whose direction is perpendicular to the crack front. The problem of the plane deformation of a prestressed plane with a slit is equivalent to the problem of integrating system (3) in the half-plane $y \geqslant 0$, with the following mixed boundary conditions at the boundary $y = 0$ of the half-plane:

$$q = 0,\ |x| < \infty \tag{13}$$

$$\theta_{21} = \nu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_0 = \text{const},\quad |x| < a \tag{14}$$
$$u = 0,\ |x| > a$$

Using the method of paired integral equations we obtain the solution of problem (3), (13), (14), in the following form:

$$u = -\tau_0\nu^{-1}\Phi\omega_1\omega_2\,\mathrm{Re}\,\{(\omega_1 - \omega_2)^{-1}[(\nu - \mu + \nu\omega_2^2)(\sqrt{a^2 - z_1^2} + iz_1) - \ldots]\}$$

$$v = \tau_0\nu^{-1}\Phi\,\mathrm{Im}\,\{(\omega_1 - \omega_2)^{-1}[\omega_2(\nu - \mu + \nu\omega_2^2)(\sqrt{a^2 - z_1^2} + iz_1) - \ldots]\}$$

$$q = \tau_0\nu^{-1}\omega_1\omega_2(\nu - \mu + \nu\omega_1^2)(\nu - \mu + \nu\omega_2^2)\Phi\,\mathrm{Im}\left\{\left[(\omega_1 - \omega_2)^{-1}\left(\frac{iz_1}{\sqrt{a^2 - z_1^2}} - \ldots\right]\right\}\right.$$

$$\theta_{21} = \tau_0\Phi\,\mathrm{Re}\left\{(\omega_1 - \omega_2)^{-1}\left[\omega_2(\nu - \mu + \nu\omega_2^2)(1 + \omega_1^2)\left(1 + \frac{iz_1}{\sqrt{a^2 - z_1^2}}\right) - \ldots\right]\right\}$$

$$\theta_{12} = \tau_0\nu^{-1}\Phi\,\mathrm{Re}\left\{(\omega_1 - \omega_2)^{-1}\left[\omega_2(\nu - \mu + \nu\omega_2^2) \times (\varkappa + \nu\omega_1^2)\left(1 + \frac{iz_1}{\sqrt{a^2 - z_1^2}}\right) - \ldots\right]\right\}$$

$$\theta_{11} = \tau_0\Phi\omega_1\omega_2\,\mathrm{Im}\left\{(\omega_1 - \omega_2)^{-1}\left[(\nu - \mu + \nu\omega_2^2)(1 + \omega_1^2)\left(1 + \frac{iz_1}{\sqrt{a^2 - z_1^2}}\right) - \ldots\right]\right\}$$

(the quantities $z_1, z_2$ are given by Eqs.(7)). On the extension of the crack line we have, according to (15),

$$u|_{y=+0} = \begin{cases} \tau_0\Phi\omega_1\omega_2(\omega_1+\omega_2)\sqrt{a^2-x^2}, & |x|\leqslant a \\ 0, & |x|\geqslant a \end{cases} \tag{16}$$

$$v|_{y=0} = \tau_0\Phi[\mu\nu^{-1} - 1 + \omega_1\omega_2 + \omega_1^2 + \omega_2^2]\begin{cases} x, & |x|\leqslant a \\ (x-\sqrt{x^2-a^2}), & x\geqslant a \end{cases}$$

$$\theta_{21}|_{y=0} = \begin{cases} -\tau_0, & |x|<a \\ \tau_0(|x|/\sqrt{x^2-a^2}-1), & |x|>a \end{cases}$$

We note that the horizontal displacements of different slit edges have different signs, while the vertical displacements are the same at both edges.

Formulas (15) show that in the problem of transverse shear the stress singularities at the crack tip are of the same order as in the analogous problem without the initial stresses /3/.

A detailed pattern of displacement and stress distribution near the singularity $x=a$ can be obtained by substituting relations (11), (12) into (15).

Let us consider in detail the case of Mooney material under an initial plane deformation, for which we have

$$\begin{gathered} \varkappa = G\lambda^2,\ \nu = G\lambda^{-2},\ \mu = G(\lambda^2+3\lambda^{-2}) \\ \lambda_3 = 1,\ \lambda_1 = \lambda = \lambda_2^{-1},\ \omega_1 = 1,\ \omega_2 = \lambda^2 \\ t_{11} = G(\lambda-\lambda^{-2}),\ t_{22} = t_{12} = 0,\ G = 2(C_1+C_2) \\ \Phi = \frac{\lambda^2}{GK(\lambda)},\quad K(\lambda) = \lambda^6+\lambda^4+3\lambda^2-1 \end{gathered} \tag{17}$$

Here $C_1$, $C_2$ are the elastic constants of the Mooney material.
Eqs.(16) for the Mooney material will be

$$u|_{y=+0} = \frac{\tau_0\lambda^4(1+\lambda^2)}{GK(\lambda)}\sqrt{a^2-x^2},\quad |x|\leqslant a \tag{18}$$

$$v|_{y=0} = \tau_0\frac{\lambda^2(1-\lambda^2)}{GK(\lambda)}\begin{cases} x, & |x|\leqslant a \\ (x-\sqrt{x^2-a^2}), & x\geqslant a \end{cases}$$

As $\lambda\to 1$, relations (18) become the well-known expressions for the displacements of the crack edges in the analogous problem without initial stresses /3/

$$u|_{y=+0} = \frac{\tau_0}{2G}\sqrt{a^2-x^2},\quad v|_{y=0} = 0,\quad |x|\leqslant a$$

Thus, taking the initial stresses into account leads to the appearance of vertical displacements of the crack edges under a tangential shearing load. The displacements do not lead to an opening of the crack, since they are the same for both edges.

The equation $K(\lambda)=0$ has a unique real root $\lambda^* = 0.545$. When $\lambda\to\lambda^*$, the displacements increase without limit at the crack surface. This means that when $\lambda\leqslant\lambda^*$, the homogeneous stress-strain state of a compressed plane with a crack is unstable.

Using (15) we derive the following asymptotic representations of the solution near the singularity $x=a$:

$$u = \frac{\tau_0 a\lambda^6}{GK(\lambda)(1-\lambda^2)}\left[2\lambda^{-2}\sqrt{2\rho}\sin\frac{\varphi}{2} - (\lambda^2+\lambda^{-2})\sqrt{2\rho_2}\sin\frac{\varphi_2}{2}\right] + O(\rho) \tag{19}$$

$$v = \frac{\tau_0 a\lambda^4}{GK(\lambda)(1-\lambda^2)}\left[2\sqrt{2\rho}\cos\frac{\varphi}{2} - (\lambda^2+\lambda^{-2})\sqrt{2\rho_2}\cos\frac{\varphi_2}{2} + (\lambda-\lambda^{-1})^2\right] + O(\rho)$$

$$\theta_{21} = \frac{\tau_0}{K(\lambda)(1-\lambda^2)}\left[\frac{4\lambda^2}{\sqrt{2\rho}}\cos\frac{\varphi}{2} - \frac{(1+\lambda^4)^2}{\sqrt{2\rho_2}}\cos\frac{\varphi_2}{2}\right] + O(1)$$

$$\theta_{22} = \sigma_{22} = \Lambda\left(\frac{1}{\sqrt{2\rho}}\sin\frac{\varphi}{2} - \frac{1}{\sqrt{2\rho_2}}\sin\frac{\varphi_2}{2}\right) + O(\sqrt{\rho})$$

$$\theta_{12} = \sigma_{12} = 2\Lambda\left(\frac{1}{\sqrt{2\rho}}\cos\frac{\varphi}{2} - \frac{\lambda^2}{\sqrt{2\rho_2}}\cos\frac{\varphi_2}{2}\right) + O(1)$$

$$\sigma_{11} = 2\Lambda\left(\frac{1}{\sqrt{2\rho}}\sin\frac{\varphi}{2} - \frac{\lambda^4}{\sqrt{2\rho_2}}\sin\frac{\varphi_2}{2}\right) + O(1)$$

$$\Lambda = \frac{\tau_0\lambda^2(1+\lambda^4)}{K(\lambda)(1-\lambda^2)},\quad \rho_2 = \rho\Delta,\quad \Delta = (\cos^2\varphi + \lambda^4\sin^2\varphi)^{1/2}$$

$$\sin\frac{\varphi_2}{2} = \frac{\operatorname{sign}\varphi}{\sqrt{2}}\left(1-\frac{\cos\varphi}{\Delta}\right)^{1/2},\quad \cos\frac{\varphi_2}{2} = \frac{1}{\sqrt{2}}\left(1+\frac{\cos\varphi}{\Delta}\right)^{1/2}$$

It can be shown that when the initial stresses are removed, i.e. as $\lambda \to 1$, formulas (19) become the well-known /3/ asymptotic expressions for the displacements and stresses near the crack tip in the corresponding problem without initial stresses.

In the problem of the longitudinal shear of a crack we consider an unbounded, elastic predeformed space containing a plane slit whose edges coincide with the straight lines $y = 0$, $x = a$ and $y = 0, x = -a$. The slit surfaces are uniformly loaded in the direction along the crack edges, and the directions of the loads acting on opposite sides of the cut (slit) have opposite signs.

If the principal axes of the initial deformation coincide with the coordinate axes and the body is isotropic, then using Eqs.(1), (2) we can conclude that the additional deformation appearing when the crack is under such a load will be antiplane. This means that

$$u = v = 0,\ u_3 = w(x, y)$$

The equations of equilibrium reduce, in the problem of antiplane deformation of a prestressed body, to the single equation

$$\frac{\partial \theta_{13}}{\partial x} + \frac{\partial \theta_{23}}{\partial y} = 0, \quad \theta_{13} = \pi_1 \frac{\partial w}{\partial x}, \quad \theta_{23} = \pi_2 \frac{\partial w}{\partial y} \tag{20}$$

$$\pi_1 = \lambda_1^2 \frac{\lambda_3 \Pi_3 - \lambda_1 \Pi_1}{\lambda_3^2 - \lambda_1^2}, \quad \pi_2 = \lambda_2^2 \frac{\lambda_2 \Pi_2 - \lambda_3 \Pi_3}{\lambda_2^2 - \lambda_3^2}$$

The problem of a crack formulated above can be replaced by the equivalent problem for a half-plane $y \geqslant 0$ with the following boundary conditions on the straight line $y = 0$:

$$\theta_{23} = -t_0 = \text{const},\ |x| < a;\ w = 0,\ |x| > a \tag{21}$$

Applying the integral Fourier transform we arrive at a paired integral equation whose solution yields an expression for a displacement in the upper half-plane

$$w = \frac{t_0}{\sqrt{\pi_1 \pi_2}} \operatorname{Re}\left(\sqrt{a^2 - z_3^2} + i z_3\right), \quad z_3 = x + i\omega_3 y, \quad \omega_3 = \sqrt{\frac{\pi_1}{\pi_2}} \tag{22}$$

For the displacement at the slit edge we have

$$w = \pm \frac{t_0 a}{\sqrt{\pi_1 \pi_2}} \sqrt{1 - \frac{x^2}{a^2}}, \quad |x| \leqslant a \tag{23}$$

where the upper sign refers to the upper edge, and the lower sign to the lower edge.

The tangential stress on the line continuing the crack is given by

$$\theta_{23} = -t_0 \left(1 - \frac{|x|}{\sqrt{x^2 - a^2}}\right), \quad |x| > a \tag{24}$$

It can be shown that in the present problem the additional true stresses are given by

$$\sigma_{13} = \theta_{13},\ \sigma_{23} = \theta_{23},\ \sigma_{\alpha\beta} = \sigma_{33} = 0 \quad (\alpha, \beta = 1, 2)$$

The asymptotic expression for the displacement and stress field diestribution about the crack edge is given, in the problem of longitudinal shear, by the formulas

$$w = \frac{t_0 a}{\sqrt{\pi_1 \pi_2}} \sqrt{2\rho_3} \sin \frac{\varphi_3}{2} + O(\rho) \tag{25}$$

$$\sigma_{13} = -\frac{t_0}{\sqrt{\pi_2}} \sqrt{\pi_1}\, (2\rho_3)^{-1/2} \sin \frac{\varphi_3}{2} + O\left(\sqrt{\rho}\right)$$

$$\sigma_{23} = t_0 (2\rho_3)^{-1/2} \cos \frac{\varphi_3}{2} + O(1)$$

The quantities $\varphi_3$ and $\rho_3$ are given by relations (12) in which we must put $k = 3$.

From the solution obtained above it follows that in case of a longitudinal shear of a prestressed body with a crack, the order of the stress singularity at the crack edge is the same as in the case without the initial stresses. The initial stresses affect the form of the stress distribution around the singularity.

We will now consider the question of what effect the prior loading has on the stress intensity coefficients, which play an important part in determining the strength in the case of brittle fracture in solids with cracks.

In the linear theory of elasticity the stress intensity coefficients are brought in as follows /3/. Consider a point $O$ on the contour of a plane crack of arbitrary shape. We introduce a local Cartesian coordinate system with origin at the point $O$, so that the $x$ axis is orthogonal to the crack contour, the $y$ axis is perpendicular to the plane of the crack, and the $z$ axis is directed along the tangent to the crack boundary. The normal fracture $\sigma_{yy}$, transverse shear $\sigma_{xy}$ and longitudinal shear $\sigma_{yz}$ stresses have, for any load applied to the body, the following form at points of the $x$ axis situated near the point $O$:

$$\sigma_{yy} = k_1 (2x)^{-1/2} + O(1), \quad \sigma_{xy} = k_2 (2x)^{-1/2} + O(1)$$
$$\sigma_{yz} = k_3 (2x)^{-1/2} + O(1)$$

The quantities $k_1, k_2, k_3$ are called the stress intensity coefficients. They depend on the choise of the point on the crack contour, on the form of the body, and on the load applied.

Formulas (9) and (24) show that when the incompressible body has initial stresses, the additional stresses of normal fracture and longitudinal shear on the extension of the crack line have, in the plane problem of a rectilinear crack under a uniform load applied along the edges, the same expression as those in the analogous problem without the initial stresses. This implies that the initial deformations have no effect on the stress intensity coefficients in the case of normal fracture and longitudinal shear. The same conclusion holds in the case of a circular crack acted upon by a uniform normal load /8/.

In the problem of transverse shear the quantity $\theta_{21}$ on the extension of the line of the crack is, according to (16), the same as the expression for the tangential stress $\sigma_{12}$ in the classical theory of elasticity. The additional transverse shear stress in a prestressed body, however, has, according to (1), the form

$$\sigma_{12} = \theta_{21} + t_{11}\partial v/\partial x$$

and differs from the tangential stress on the extension of the line of the crack in the corresponding problem of the linear theory of elasticity. For this reason, the transverse shear stress intensity coefficient, when the crack surface is uniformly loaded, depends on the initial deformation.

For Mooney material, in particular, we have, according to (17), when $y = 0$ in the neighbourhood of the crack tip,

$$\sigma_{12} = \frac{t_0}{\sqrt{2\rho}}\left[1 + \frac{(1+\lambda^2)(1-\lambda^2)^2}{K(\lambda)}\right] + O(1)$$

Since the quantity $K(\lambda)$ is positive in the admissible interval $\lambda^* < \lambda < \infty$, the last formula shows that the presence of initial stresses (irrespective of their signs), results in an increase in the transverse shear stress intensity coefficient.

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